Mean-field theory for car accidents

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We study analytically the occurrence of car accidents in the Nagel-Schreckenberg traffic model. We obtain exact results for the occurrence of car accidents P_{ac} as a function of the car density ρ and the degree of stochastic braking p_1 in the case of speed limit $v_{\max}=1$. Various quantities are calculated analytically. The nontrivial limit $p_1 \to 0$ is discussed.

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I. INTRODUCTION

The cellular automaton models of traffic flow have attracted much interest recently [1]. Instead of differential equations, the underlying dynamics is governed by a few simple update rules. They allow the flexibility to adapt complicated features observed in real traffic and can be used very efficiently for computers to perform real-time simulations. Numerical works in numerous applications have been reported [2]. In contrast, we know very little about the analytical properties. As one does not have a Hamiltonian description, standard methods in statistical mechanics are not applicable. Not much has been known about the exact solutions of discrete-time update. The use of parallel update introduces further difficulties for the strong correlations involved. There is a need for exact solutions that may provide better insight to the models and greatly help to reduce the need for computer resources.

More recently, the occurrence of car accidents in a cellular automaton model has been studied numerically in a special case [3]. Later, general results are reported [4]. Basically, there are two parameters: speed limit $v_{\max}$ and braking probability p_1, which will be defined more specifically in the next section. The exact results have been obtained in a special case [5]. Only the analytical properties of the speed limit $v_{\max}$ have been studied. The effects of braking probability p_1 are totally neglected, i.e., $p_1=0$. In this paper, we study analytically the effects of braking probability p_1. Exact solutions are obtained for the occurrence of car accidents in a traffic model. An interesting limit $p_1 \to 0$ is discussed.

II. CAR ACCIDENTS

The Nagel-Schreckenberg model is a basic model of traffic flow on a single-lane highway [6]. Both the space and time are discretized. The road is divided into discrete cells. Each cell can be either empty or occupied by a car. With prescribed rules, the motion of these cars is determined by updating the configuration at discrete time steps. The model has two parameters: the speed limit $v_{\max}$ and the braking probability p_1, which are applied to all cars. For each car, the speed is also an integer $v \in \{0,1,\ldots,v_{\max}\}$, which is mainly determined by the distance to the car ahead. When the distance increases, the car accelerates; when the distance decreases, the car slows down. The configuration is then updated by the following four specific rules sequentially.

(1) Acceleration: If $v<v_{\max}$, then $v \to v+1$. (2) Slowing down: If $v>d$, then $v \to d$. (3) Randomization: If $v>0$, then $v \to v-1$ with probability p_1. (4) Motion: The position of a car is shifted by its speed v.

The number of empty cells in front of a car is denoted by d. The first three rules adjust the speed of a car, which is then applied in the fourth rule. The acceleration under the speed limit and the slowing down due to the car ahead are prescribed by the first two rules. Without the third rule, the model is deterministic. The third rule introduces a noise to simulate the stochastic driving behavior. These update rules are applied in parallel to all cars. Iterations over these simple rules already give realistic results.

In the basic model, car accidents will not occur. The second rule of the update is designed to avoid accidents; the driving scheme respects the safety distance. In real traffic, car accidents occur most likely when drivers do not respect the safety distance, which often happens when the car ahead is moving. If a moving car is suddenly stopped, a careless driving of the following car will result in an accident. Thus the occurrence of car accidents can be associated with the following three conditions simultaneously satisfied: (1) $d \leqslant v_{\max}$, (2) $v'>0$ at t, (3) $v'=0$ at $t+1$.

The speed of the car ahead is denoted by v'. The first condition implies two cars are near, i.e., the position of the car ahead can be reached by the next time step. Otherwise, a car accident is not likely to occur. The last two conditions require a moving car ahead and its sudden stop at the next time step. The simultaneous satisfaction of these three conditions describes a dangerous situation on road. The occurrence of car accidents is expected to be proportional to the occurrence of such dangerous situations. The proportional constant is denoted by p_2, i.e., when these three conditions are satisfied, a car will cause an accident with a probability p_2. In the numerical simulations, the car accident defined as a car that hits the car ahead, does not really happen. We are looking for those dangerous situations on the road and take them as the indicator to the occurrence of car accidents. The probability *per car* and *per time step* for an accident to occur is denoted by P_{ac}. As P_{ac} is proportional to p_2, we will study the quantity P_{ac}/p_2 and leave the probability p_2 unspecified. In Refs. [3,4], the function $P_{ac}(\rho)$ is studied numerically with two parameters: $v_{\max}$ and p_1. The analytical properties of $v_{\max}$ has been reported in [5]. In the next section, we will study the analytical properties of p_1. The exact results in the case of $v_{\max}=1$ will be obtained.

III. MEAN-FIELD THEORY

The simplest analytical approach is a microscopic mean-field theory, where correlations between cells are completely neglected. With $v_{\max}=1$, there are three configurations for a single cell. Three variables are employed: Q_i with $i \in \{x,0,1\}$, and denote the probability to find an empty cell, a stopped car (a car with speed 0), and a moving car (a car with speed 1), respectively. The normalization of probability gives

$$Q_x+Q_0+Q_1=1. \quad (1)$$

As the number of cars is conserved, we have

$$Q_0+Q_1=\rho. \quad (2)$$

The master equation for the stationary state leads to

$$Q_0[(1-p_1)Q_x]=Q_1[p_1Q_x+Q_0+Q_1]. \quad (3)$$

The left-hand side gives the decreasing of Q_0 within a time step; the right-hand side gives its increasing. The cell with a stopped car will change its configuration if the car has an empty cell in front of it (a factor Q_x) and the braking is not applied [a factor $(1-p_1)$]. On the contrary, a moving car will be stopped in three different situations. The first one is that the car has an empty cell in front of it (a factor Q_x) but the braking is applied (a factor p_1). The second one is that the car has a stopped car ahead (a factor Q_0). The third one is that the car has a moving car ahead (a factor Q_1). We note that the third situation is a characteristic for the parallel update. The balancing between the decreasing and increasing of Q_1 leads to the same equation. Thus the following analytical expressions can be solved

$$Q_x=1-\rho, \quad (4)$$

$$Q_0=\rho-(1-p_1)\rho(1-\rho), \quad (5)$$

$$Q_1=(1-p_1)\rho(1-\rho). \quad (6)$$

With these three variables, an analytical expression for the probability P_{ac} can be obtained as

$$P_{ac}=\frac{p_2}{\rho}(1+Q_x)(Q_1)(Q_0), \quad (7)$$

where the factors in three parentheses correspond to the three conditions of accidents, respectively. The first condition requires the number of empty cells (in front of the car) to be zero or one. The second condition requires a moving car (the car ahead). The third condition requires a stopped car (the car ahead in the next time step). The result are shown in Fig. 1. As both the spatial and temporal correlations are neglected, this formula is not expected to give an accurate description. The simple mean-field theory overestimates the value of P_{ac} considerably in the low-density region, which can be related to the underestimation of the flow. We note that the apparent symmetry with respect to $\rho=0.5$ in Fig. 1 is an artifact of $p_1=0.5$. On the contrary, the same symmetry in the fundamental diagram (flow vs density) is a general feature due to ''particle-hole'' symmetry (which means that driving a car to the right is the same as driving an empty cell to the left) and is valid for any value of p_1.

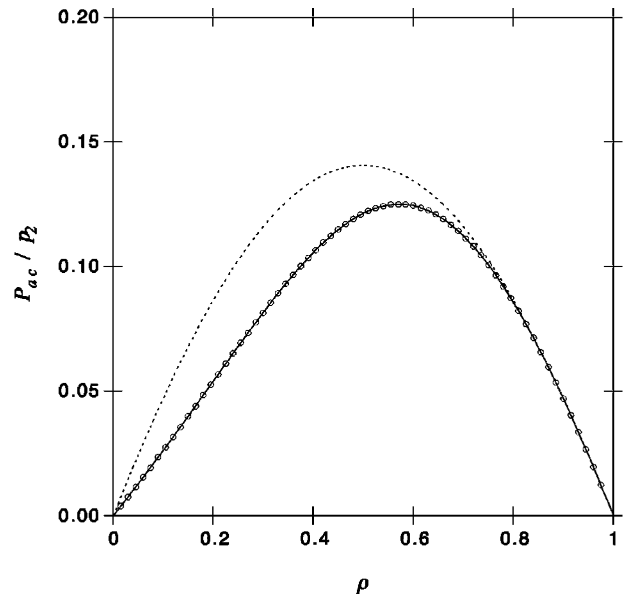

FIG. 1. Probability P_{ac} (scaled by p_2) as a function of density ρ for $p_1=0.5$. The data points are the results of numerical simulations. The dashed line is the simple mean-field result. The solid line is the mean-field result with nearest-neighbor correlation.

The short-range correlations between the cells can be taken into account in a systematic improvement of the mean-field theory [7]. Considering the nearest-neighbor correlations, there are nine variables: Q_{ij} with $i,j \in \{x,0,1\}$, which describe the configurations of two nearest-neighboring cells. Similarly, the values of Q_{ij} can be solved analytically. The normalization gives

$$Q_{xx}+Q_{x0}+Q_{x1}+Q_{0x}+Q_{00}+Q_{01}+Q_{1x}+Q_{10}+Q_{11}=1. \quad (8)$$

The conservation of density gives

$$Q_{x0}+Q_{x1}+Q_{0x}+2Q_{00}+2Q_{01}+Q_{1x}+2Q_{10}+2Q_{11}=2\rho. \quad (9)$$

The stationary probabilities are determined by the dynamics of the update rules. It is interesting to note that the parallel update implies that two of the nine variables vanish, i.e., $Q_{01}=Q_{11}=0$. A moving car must leave an empty cell behind. These two forbidden configurations ($\{01\}$ and $\{11\}$) are also known as the Garden of Eden states of the dynamics [8]. The equations for the stationary probabilities can be obtained by the combination of conditional probabilities. For example, the equation for Q_{xx} reads

$$\left[\frac{(1-p_1)Q_{0x}+(1-p_1)Q_{1x}}{Q_{xx}+Q_{0x}+Q_{1x}}\right]Q_{xx} =\left[\frac{Q_{xx}+p_1Q_{0x}+p_1Q_{1x}}{Q_{xx}+Q_{0x}+Q_{1x}}\right]Q_{x0}\left[\frac{(1-p_1)Q_{0x}}{Q_{0x}+Q_{00}}\right] +\left[\frac{Q_{xx}+p_1Q_{0x}+p_1Q_{1x}}{Q_{xx}+Q_{0x}+Q_{1x}}\right]Q_{x1}\left[\frac{(1-p_1)Q_{1x}}{Q_{1x}+Q_{10}}\right]. \quad (10)$$

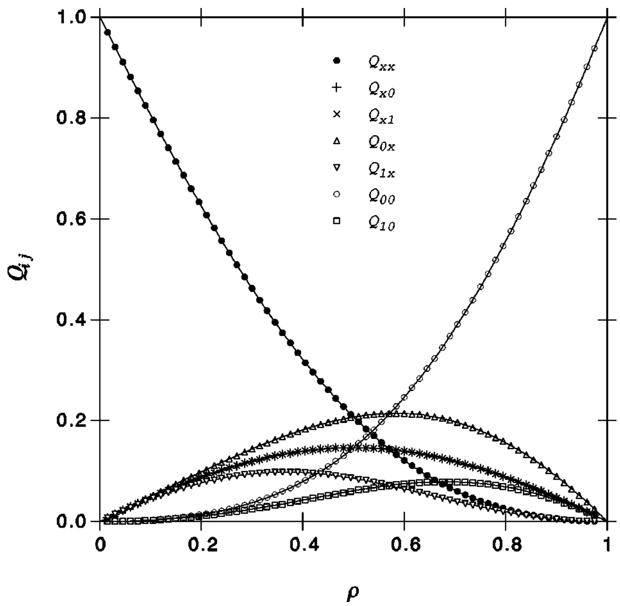

FIG. 2. Probability Q_{ij} as a function of density ρ for $p_1 = 0.5$. The data points are the results of numerical simulations. The solid lines are the mean-field results with nearest-neighbor correlations.

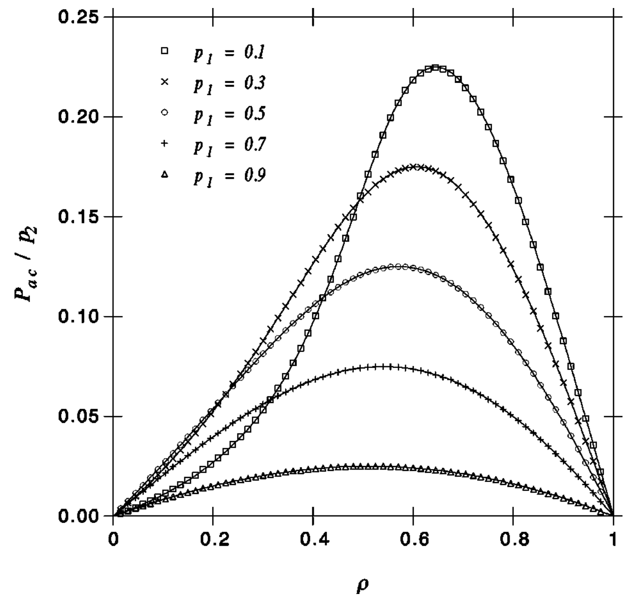

FIG. 3. Probability P_{ac} (scaled by p_2) as a function of density ρ for various p_1. The data points are the results of numerical simulations. The solid lines are the mean-field results with nearest-neighbor correlations.

The left-hand side gives the decreasing of Q_{xx}; the right-hand side gives its increasing. There are six more equations for $Q_{x0}, Q_{x1}, Q_{0x}, Q_{00}, Q_{1x}$, and Q_{10}, which are not shown. These equations are not linearly independent of each other. However, together with Eqs. [8] and [9], they provide a unique solution for Q_{ij}. For example, the solution for Q_{xx} reads

$$Q_{xx} = 1 - \rho - \frac{1 - \sqrt{1 - 4(1-p_1)\rho(1-\rho)}}{2(1-p_1)}. \tag{11}$$

Similar expressions for $Q_{x0}, Q_{x1}, Q_{0x}, Q_{00}, Q_{1x}$, and Q_{10} are obtained, which are shown in the Appendix. These analytical expressions provide the exact results, see Fig. 2.

Next, the value of P_{ac} can be rewritten as

$$P_{ac} = \frac{p_2}{\rho}(Q_{0x} + Q_{1x})\left(\frac{Q_{x1}}{Q_{xx} + Q_{x0} + Q_{x1}}\right)\left(\frac{Q_{10} + p_1 Q_{1x}}{Q_{10} + Q_{1x}}\right), \tag{12}$$

where the factors in three parentheses still correspond to the three conditions of accidents, respectively, and the combination of conditional probabilities has been applied. We note that, as required in the first condition, the number of empty cells must be one. Since a moving car must leave an empty cell behind, the case without an empty cell is excluded. We further note that the temporal correlation in the third condition is taken into account in the third parenthesis. For a moving car to stop in the next time step, the moving car is prescribed to either have a stopped car ahead or have an empty cell in front of it but the braking is applied (with probability p_1). Again, the moving car cannot have another moving car ahead, i.e., $Q_{11} = 0$. As expected, this formula gives an exact result for the value of P_{ac}, see Fig. 3. With the analytical expressions of Q_{ij}, the above equation can be rewritten as

$$P_{ac} = p_2 \frac{[(1-p_1)(3\rho - 2\rho^2) - 1] - [(1-p_1)\rho - 1]\sqrt{1 - 4(1-p_1)\rho(1-\rho)}}{2(1-p_1)\rho^2}. \tag{13}$$

IV. DISCUSSION

In summary, we obtain exact results for the occurrence of car accidents P_{ac} in the asymptotic steady state as a function of the car density ρ and the degree of stochastic braking p_1 in the Nagel-Schreckenberg traffic model with $v_{\max} = 1$. The analytical approach is based on a phenomenological mean-field theory with nearest-neighbor correlations. Both the spatial and temporal correlations have been considered. As the density ρ increases the value of P_{ac} increases, reaches a maximum, and then decreases with further increase of ρ. As the braking probability p_1 increases, the value of P_{ac} is enhanced in the low-density region and suppressed in the high-density region. With the exact solution in Eq. (13), various quantities can be calculated analytically. For example, in the high-density region, we have

$$P_{ac} \sim p_2(1-p_1)(1-\rho). \tag{14}$$

Conservative driving (with a large stochastic braking prob-

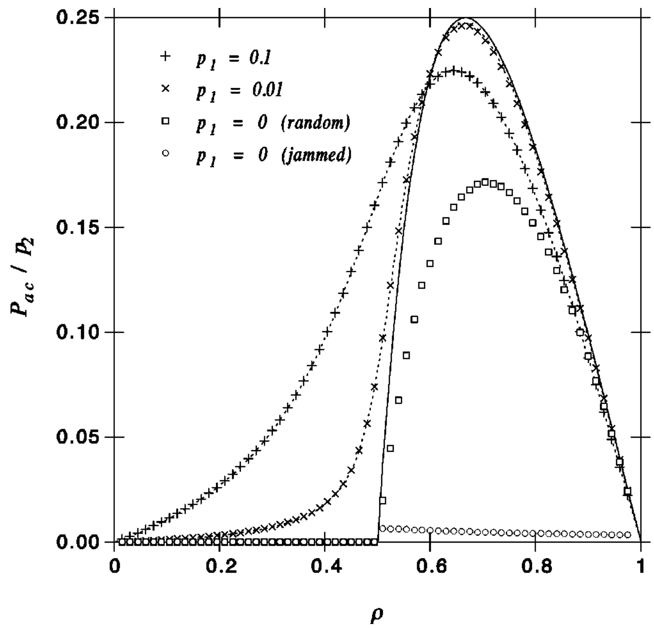

FIG. 4. Probability P_{ac} (scaled by p_2) as a function of density ρ for various p_1. The dashed lines are the exact results for $p_1=0.1$ and 0.01. The solid line is the limiting case of $p_1\to 0$. The data points are the results of numerical simulations. The circles are the results for $p_1=0$ with random initial configurations. The squares are the results for $p_1=0$ with jammed initial configurations.

ability) does reduce the occurrence of accidents. This is not true in the low-density region, where we have

$$P_{ac}\sim p_2 p_1(1-p_1)\rho. \tag{15}$$

The occurrence of accidents is also reduced in the case of aggressive driving (with a small stochastic braking probability). The value of P_{ac} increases with the increase of density ρ at a maximum slope in the case of $p_1=0.5$. The maximum of P_{ac} can also be calculated analytically as

$$P_{ac}(\rho')=[p_2(1-p_1)]/4\,, \tag{16}$$

where the corresponding density is

$$\rho'=2/(3+p_1)\,. \tag{17}$$

As the phenomena of car accidents involves strong correlations, the success of phenomenological mean-field theory is encouraging. However, the success cannot be extended to the case of $v_{\max}>1$, where only the approximate results are obtained.

We also find that the limit $p_1\to 0$ is nontrivial. With $p_1=0$, Eq. (13) reduces to

$$P_{ac}=\begin{cases} 0 & \text{for } \rho<0.5,\\ p_2\dfrac{(2\rho-1)(1-\rho)}{\rho^2} & \text{for } \rho\geqslant 0.5.\end{cases} \tag{18}$$

However, this formula does not describe the numerical results. With $p_1>0$, the dynamics are stochastic. The stationary probabilities Q_{ij} and P_{ac} are independent of the initial configurations. The solution is unique, as shown in Eq. (13). With $p_1=0$, the dynamics becomes deterministic. Some of the equations become identity, and the rest of them do not provide a unique solution. The stationary probabilities Q_{ij} and P_{ac} depend on the initial configurations. Different initial configurations will lead to different results, see Fig. 4. If we start with a random configuration, the exact results are [5]

$$P_{ac}=\begin{cases} 0 & \text{for } \rho<0.5,\\ p_2\dfrac{(2\rho-1)(1-\rho)}{\rho} & \text{for } \rho\geqslant 0.5.\end{cases} \tag{19}$$

If we start with a jammed configuration, the value of P_{ac} is greatly reduced. In the asymptotic steady state, the occurrence of accidents becomes a finite-size effect. In such case, there will be no accidents in the limit $L\to\infty$.

APPENDIX

To be complete, we list the analytical expressions for the six variables $Q_{x0}, Q_{x1}, Q_{0x}, Q_{00}, Q_{1x}$, and Q_{10} in the following:

$$Q_{x0}=\frac{p_1}{1-p_1}\mathcal{X},$$

$$Q_{x1}=\mathcal{X},$$

$$Q_{0x}=\frac{\mathcal{X}}{1-p_1}-\frac{\mathcal{X}^2}{(1-p_1)\rho},$$

$$Q_{00}=\rho-\mathcal{X}-\frac{\mathcal{X}}{1-p_1}+\frac{\mathcal{X}^2}{(1-p_1)\rho},$$

$$Q_{1x}=\frac{\mathcal{X}^2}{(1-p_1)\rho},$$

$$Q_{10}=\mathcal{X}-\frac{\mathcal{X}^2}{(1-p_1)\rho},$$

where $\mathcal{X}\equiv\frac{1}{2}[1-\sqrt{1-4(1-p_1)\rho(1-\rho)}]$. Comparison to the numerical results are shown in Fig. 2. In the special case of $p_1=0.5$, we have $Q_{x0}=Q_{x1}$.

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